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Toffoli gate made from a single resonant interaction with a trapped ion system

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Abstract. We propose a simple but practical scheme to implement a three-qubit Toffoli gate by a single resonant interaction in a trapped ion system. The scheme does not require two-qubit controlled-NOT gates but uses a three-qubit phase gate and two Hadamard gates, where the phase gate can be implemented by only a single resonant interaction of the trapped ions with the first lower vibrational sideband mode. Both the situations, with and without spontaneous ionic emission, are investigated. Discussions are made for the advantages and the experimental feasibility of our scheme.

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Shor's discovery of a polynomial-time quantum factoring algorithm [1], and Grover's search [2] for an item with a quantum computer from an unsorted list system, have directly induced the extensive studies on quantum information processing. Although the building blocks of quantum computers are one- and two-qubit logic gates [3], universal multi-qubit operations of the form $C^n - NOT$ $(n \geqslant 2)$ are very useful for quantum computation [4] because a direct implementation of $C^n - NOT$ gates requires a shorter gating time than implementations consisting of a series of one- and two-qubit operations. In the NMR system, some general methods for creating $C^n - NOT$ gates have been proposed [5]. Recently, a scheme for the realization of conditional quantum gates has been proposed, including a N-atom Toffoli gate and some other nonlocal gates on remote atoms, through cavity-assisted photon scattering [6], in which single photon detectors and a series of linear optical elements are necessary. An alternative scheme for the three-qubit Toffoli gate could be found in [7], based on vacuum-induced Stark shifts using a collective system in a high-quality dispersive cavity. Nevertheless, the Toffoli gate in a trapped ion system has not yet been achieved experimentally.

In this paper, we propose a practical scheme to implement a three-qubit Toffoli gate by only a single resonant interaction in a trapped ion system. The scheme does not require two-qubit controlled-NOT gates but uses a three-qubit phase gate and two Hadamard gates to construct the Toffoli gate. We will focus on a phase gate implemented by using laser beams to excite the trapped ions simultaneously. Our scheme is quite simple because it does not use the vibrational mode as the data bus and only requires one resonant interaction. Also the required interaction time is very short due to the resonant interaction. Moreover, our scheme is within the reach of current cavity trapped ion techniques.

First, the resonant interaction of N three-level trapped ions with several laser beams is considered. The ionic internal states can be expressed by $|i_j\rangle$, $|g_j\rangle$, and $|e_j\rangle$, with $|g_j\rangle$ and $|i_j\rangle$ being states lower than $|e_j\rangle$. In our scheme, the states $|i_j\rangle$ are not involved in the interaction with the vibrational mode. The computational basis is spanned by the atomic states $\{|g_1\rangle, |e_1\rangle, |g_k\rangle, |i_k\rangle, k = 2, ..., N\}$. Assume that the N ions are confined in a linear trap and then each ion is excited by a laser. All lasers are tuned to the first lower vibrational sideband. In the Lamb-Dicke limit, the Hamiltonian in units of $\hbar = 1$ is

$$H_i = \sum_{j=1}^{N} i\eta\Omega_j (a^+ S_j^- e^{i\phi_j} - a S_j^+ e^{-i\phi_j}), \tag{1}$$

where Ω_j and ϕ_j $(j = 1, \ldots N)$ are the Rabi frequencies and phases of the laser fields, and $S_j^+ = |e_j\rangle\langle g_j|$ and $S_j^- = |g_j\rangle\langle e_j|$ are the ionic spin operators for flipping

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the system between states, respectively. a^+ (a) is the creation (annihilation) operator for the first lower vibrational mode. Furthermore, by assuming that $\phi_j = -\pi/2$, equation (1) reduces to

$$H_i = \sum_{j=1}^{N} \eta \Omega_j (a^+ S_j^- + a S_j^+). \tag{2}$$

If the ions and the vibrational mode are initially in the state $\prod_{j=2}^{N} |e_1\rangle |g_j\rangle$ and the vacuum state $|0\rangle$, respectively. The evolution of the system is obtained straightforwardly by,

$$\begin{aligned} |\psi(t)\rangle &= U(t) \prod_{j=2}^{N} |e_1\rangle\, |g_j\rangle\, |0\rangle \\ &= \exp[-it \sum_{j=1}^{N} \eta \Omega_j (a^+ S_j^- + a S_j^+)] \prod_{j=2}^{N} |e_1\rangle\, |g_j\rangle\, |0\rangle \\ &= [\Omega_1^2/\Omega^2 \cos(\eta\Omega t) + (\Omega^2 - \Omega_1^2)/\Omega^2] \prod_{j=2}^{N} |e_1\rangle\, |g_j\rangle\, |0\rangle \\ &\quad + \Omega_1/\Omega^2 \{\cos(\eta\Omega t) - 1\} \sum_{k=2}^{N} \Omega_k\, |g_1\rangle\, |e_k\rangle \\ &\quad \times \prod_{j=2, j\neq k}^{N} |g_j\rangle\, |0\rangle - i\Omega_1/\Omega \sin(\eta \Omega t) \prod_{j=1}^{N} |g_j\rangle\, |1\rangle, \end{aligned} \tag{3}$$

where $\Omega = \sqrt{\sum_{j=1}^{N} \Omega_j^2}$. Before going further, let us first consider two other simple cases. If the N trapped ions are initially in the state $\prod_{j=2}^{N} |e_1\rangle |i_j\rangle$, then the ions, except the first one, do not interact with the vibrational mode. So we acquire the corresponding time evolution

$$\begin{aligned} |\psi_1(t)\rangle &= U(t) \prod_{j=2}^{N} |e_1\rangle\, |i_j\rangle\, |0\rangle \\ &= [\cos(\eta\Omega_1 t)\, |e_1\rangle\, |0\rangle - i \sin(\eta\Omega_1 t)\, |g_1\rangle\, |1\rangle] \prod_{j=2}^{N} |i_j\rangle. \end{aligned} \tag{4}$$

While if the N ions are initially in the state $\prod_{j=2, j\neq k}^{N} |e_1\rangle |g_k\rangle |i_j\rangle$, then we only have the first and the kth ions interacting with the cavity mode, as shown below,

$$\begin{aligned} |\psi_k(t)\rangle &= U(t) \prod_{j=2, j\neq k}^{N} |e_1\rangle\, |g_k\rangle\, |i_j\rangle\, |0\rangle \\ &= [\Omega_1^2/\Omega_k'^2 \cos(\eta\Omega_k' t) + (\Omega_k'^2 - \Omega_1^2)/\Omega_k'^2] \\ &\quad \times \prod_{j=2, j\neq k}^{N} |e_1\rangle\, |g_k\rangle\, |i_j\rangle\, |0\rangle \\ &\quad + \Omega_1 \Omega_k/\Omega_k'^2 [\cos(\eta\Omega_k' t) - 1]\, |g_1\rangle\, |e_k\rangle \prod_{j=2, j\neq k}^{N} |i_j\rangle\, |0\rangle \\ &\quad - i\Omega_1/\Omega_k' \sin(\eta\Omega_k' t)\, |g_1\rangle\, |g_k\rangle \prod_{j=2}^{N} |i_j\rangle\, |1\rangle, \end{aligned} \tag{5}$$

where $\Omega_k' = \sqrt{\Omega_1^2 + \Omega_k^2}$.

Based on the above four equations, we can now try to construct a three-qubit phase gate. By considering the quantum information encoded in a subspace spanned by the atomic states $\{|g_1\rangle, |e_1\rangle, |g_2\rangle, |i_2\rangle, |g_3\rangle, |i_3\rangle\}$ for $N = 3$, with the computational basis being $\{|g_1\rangle|g_2\rangle|g_3\rangle, |g_1\rangle|g_2\rangle|i_3\rangle, |g_1\rangle|i_2\rangle|g_3\rangle, |g_1\rangle|i_2\rangle|i_3\rangle, |e_1\rangle|g_2\rangle|g_3\rangle, |e_1\rangle|g_2\rangle|i_3\rangle, |e_1\rangle|i_2\rangle|g_3\rangle, |e_1\rangle|i_2\rangle|i_3\rangle\}$, we can assume that $\Omega t = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}\, t = 2l\pi/\eta$. Then we have from equation (3),

$$|e_1\rangle|g_2\rangle|g_3\rangle|0\rangle \rightarrow |e_1\rangle|g_2\rangle|g_3\rangle|0\rangle, \tag{6}$$

which means there is no change. From equation (4), by setting $\Omega_1 t = \pi/\eta$, we have

$$|e_1\rangle|i_2\rangle|i_3\rangle|0\rangle \rightarrow -|e_1\rangle|i_2\rangle|i_3\rangle|0\rangle, \tag{7}$$

which corresponds to a phase flip. For equation (5), if we consider $k = 2$, and assume $\eta\Omega_2 t = \sqrt{\Omega_1^2 + \Omega_2^2}\, t = 2m\pi/\eta$, the evolution of the state is

$$|e_1\rangle|g_2\rangle|i_3\rangle|0\rangle \rightarrow |e_1\rangle|g_2\rangle|i_3\rangle|0\rangle, \tag{8}$$

which is also unchanged. By using the above conditions for Ω, Ω_1 and Ω_2', we obtain $\sqrt{\Omega_1^2 + \Omega_3^2}\, t = \sqrt{4(l^2 - m^2) + 1}\,\pi/\eta$, which yields the evolution from equation (5) in the case of $k = 3$,

$$\begin{aligned} |\psi_3(t)\rangle &= \\ &\left[\frac{\Omega_1^2}{\Omega_1^2 + \Omega_3^2} \cos(\sqrt{4(l^2 - m^2) + 1}\pi) + \frac{\Omega_3^2}{\Omega_1^2 + \Omega_3^2}\right] |e_1\rangle|i_2\rangle|g_3\rangle|0\rangle \\ &+ \frac{\Omega_1\Omega_3}{\Omega_1^2 + \Omega_3^2} [\cos(\sqrt{4(l^2 - m^2) + 1}\pi) - 1] |g_1\rangle|i_2\rangle|e_3\rangle|0\rangle \\ &- i \frac{\Omega_1}{\sqrt{\Omega_1^2 + \Omega_3^2}} \sin(\sqrt{4(l^2 - m^2) + 1}\pi) |g_1\rangle|i_2\rangle|g_3\rangle|1\rangle. \end{aligned} \tag{9}$$

We expect to have $|e_1\rangle|i_2\rangle|g_3\rangle|0\rangle \rightarrow |e_1\rangle|i_2\rangle|g_3\rangle|0\rangle$ from the above equation, which implies $\cos(\sqrt{4(l^2 - m^2) + 1}\pi) = 1$. However, it is obvious that this condition cannot be met exactly. To satisfactorily make the best approximation to the condition, we assume $l = 5$ and $m = 3$, which yields $\cos(\sqrt{4(l^2 - m^2) + 1}\pi) = \cos(\sqrt{65}\pi) = 0.9810$. Then equation (9) reduces to

$$\begin{aligned} |\psi_3'(t)\rangle &= 0.9997 |e_1\rangle|i_2\rangle|g_3\rangle|0\rangle \\ &\quad - 0.0023 |g_1\rangle|i_2\rangle|e_3\rangle|0\rangle - i0.024 |g_1\rangle|i_2\rangle|g_3\rangle|1\rangle \\ &\simeq 0.9997 |e_1\rangle|i_2\rangle|g_3\rangle|0\rangle. \end{aligned} \tag{10}$$

This approximation will be checked thoroughly later. As the other states under consideration, including $|g_1\rangle|g_2\rangle|g_3\rangle|0\rangle$, $|g_1\rangle|g_2\rangle|i_3\rangle|0\rangle$, $|g_1\rangle|i_2\rangle|g_3\rangle|0\rangle$, and $|g_1\rangle|i_2\rangle|i_3\rangle|0\rangle$, remain unchanged in the evolution, an approximate three-qubit phase gate can be reached as follows:

$$\begin{aligned} |g_1\rangle|g_2\rangle|g_3\rangle|0\rangle &\rightarrow |g_1\rangle|g_2\rangle|g_3\rangle|0\rangle, \\ |g_1\rangle|g_2\rangle|i_3\rangle|0\rangle &\rightarrow |g_1\rangle|g_2\rangle|i_3\rangle|0\rangle, \\ |g_1\rangle|i_2\rangle|g_3\rangle|0\rangle &\rightarrow |g_1\rangle|i_2\rangle|g_3\rangle|0\rangle, \\ |g_1\rangle|i_2\rangle|i_3\rangle|0\rangle &\rightarrow |g_1\rangle|i_2\rangle|i_3\rangle|0\rangle, \\ |e_1\rangle|g_2\rangle|g_3\rangle|0\rangle &\rightarrow |e_1\rangle|g_2\rangle|g_3\rangle|0\rangle, \\ |e_1\rangle|g_2\rangle|i_3\rangle|0\rangle &\rightarrow |e_1\rangle|g_2\rangle|i_3\rangle|0\rangle, \\ |e_1\rangle|i_2\rangle|g_3\rangle|0\rangle &\rightarrow |e_1\rangle|i_2\rangle|g_3\rangle|0\rangle, \\ |e_1\rangle|i_2\rangle|i_3\rangle|0\rangle &\rightarrow -0.9997 |e_1\rangle|i_2\rangle|i_3\rangle|0\rangle, \end{aligned} \tag{11}$$

$$|\psi_{spon}(t)\rangle = U_d(t)\prod_{j=2}^{N}|e_1\rangle|g_j\rangle|0\rangle = \exp\left\{-it\left[\sum_{j=1}^{N}\eta\Omega_j(a^+S_j^- + aS_j^+) - i\frac{\Gamma}{2}\sum_{j=1}^{N}|e_j\rangle\langle e_j|\right]\right\}\prod_{j=2}^{N}|e_1\rangle|g_j\rangle|0\rangle$$
$$= \left\{\frac{\Omega_1^2}{\Omega^2}\exp\left[-\frac{\Gamma t}{4}\right]\left[\cos(A_\Gamma t) - \frac{\Gamma}{4A_\Gamma}\sin(A_\Gamma t)\right] + \frac{\Omega^2-\Omega_1^2}{\Omega^2}\exp\left[-\frac{\Gamma t}{2}\right]\right\}\prod_{j=2}^{N}|e_1\rangle|g_j\rangle|0\rangle$$
$$+ \frac{\Omega_1}{\Omega^2}\left\{\exp\left[-\frac{\Gamma t}{4}\right]\left[\cos(A_\Gamma t) - \frac{\Gamma}{4A_\Gamma}\sin(A_\Gamma t)\right] - \exp\left[-\frac{\Gamma t}{2}\right]\right\}\sum_{k=2}^{N}\Omega_k|g_1\rangle|e_k\rangle\prod_{j=2,j\neq k}^{N}|g_j\rangle|0\rangle$$
$$- i\exp\left[-\frac{\Gamma t}{4}\right]\frac{\Omega_1}{A_\Gamma}\sin(A_\Gamma t)\prod_{j=1}^{N}|g_j\rangle|1\rangle, \quad (14)$$

$$|\psi_{spon1}(t)\rangle = U_s(t)\prod_{j=2}^{N}|e_1\rangle|i_j\rangle|0\rangle$$
$$= \exp\left[-\frac{\Gamma t}{4}\right]\left\{\left[\cos(A_{1\Gamma}t) - \frac{\Gamma}{4A_{1\Gamma}}\sin(A_{1\Gamma}t)\right]\prod_{j=2}^{N}|e_1\rangle|i_j\rangle|0\rangle - i\frac{\Omega_1}{A_{1\Gamma}}\sin(A_{1\Gamma}t)\prod_{j=2}^{N}|g_1\rangle|i_j\rangle|1\rangle]\right\}, \quad (15)$$

$$|\psi_{sponk}(t)\rangle = U_s(t)\prod_{j=2,j\neq k}^{N}|e_1\rangle|g_k\rangle|i_j\rangle|0\rangle$$
$$= \left\{\frac{\Omega_1^2}{\Omega_k'^2}\exp\left[-\frac{\Gamma t}{4}\right]\left[\cos(A_{k\Gamma}t) - \frac{\Gamma}{4A_{k\Gamma}}\sin(A_{k\Gamma}t)\right] + \exp\left(-\frac{\Gamma t}{2}\right)\frac{\Omega_k'^2-\Omega_1^2}{\Omega_k'^2}\right\}\prod_{j=2,j\neq k}^{N}|e_1\rangle|g_k\rangle|i_j\rangle|0\rangle$$
$$+ \frac{\Omega_1\Omega_k}{\Omega_k'^2}\left\{\exp\left[-\frac{\Gamma t}{4}\right]\left[\cos(A_{k\Gamma}t) - \frac{\Gamma}{4A_{k\Gamma}}\sin(A_{k\Gamma}t)\right] - \exp\left(-\frac{\Gamma t}{2}\right)\right\}\prod_{j=2,j\neq k}^{N}|g_1\rangle|e_k\rangle|i_j\rangle|0\rangle$$
$$- i\exp\left[-\frac{\Gamma t}{4}\right]\eta\Omega_1/A_{k\Gamma}\sin(A_{k\Gamma}t)\prod_{j=2,j\neq k}^{N}|g_1\rangle|g_k\rangle|i_j\rangle|1\rangle, \quad (16)$$

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from which we can easily obtain a Toffoli gate in our computational subspace,

$$T = H_3T_PH_3$$
$$= \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0.00015&0.99985\\ 0&0&0&0&0&0&0.99985&0.00015 \end{bmatrix} \simeq \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{bmatrix}, \quad (12)$$

where H_3 is the Hadamard gate on the third ion and T_P is the operator for the evolution in equation (11). To achieve equation (12), we should have $\Omega_1 : \Omega_2 : \Omega_3 = 1 : \sqrt{35} : 8$ and $t = \pi/(\eta\Omega_1)$ from the above conditions for Ω, Ω_1 and Ω_3.

We now turn to study the influence of the spontaneous ionic emission on a three-qubit phase gate (or say, a three-qubit Toffoli gate). The corresponding conditional Hamiltonian [8] is,

$$H_{si} = \sum_{j=1}^{N}\eta\Omega_j(a^+S_j^- + aS_j^+) - i\frac{\Gamma}{2}\sum_{j=1}^{N}|e_j\rangle\langle e_j|, \quad (13)$$

which is a non-Hermitian and where Γ is the spontaneous emission rate. It is derived from the quantum jump approach [8]. Provided that the atoms and the collective vibrational mode are initially in the state $\prod_{j=2}^{N}|e_1\rangle|g_j\rangle$ and the vacuum state $|0\rangle$, respectively, we obtain the evolution of the system before the leakage of a photon from the trap happens,

see equation (14) above

where $\Omega = \sqrt{\sum_{j=1}^{N}\Omega_j^2}$, and $A_\Gamma = \sqrt{\sum_{j=1}^{N}\eta^2\Omega_j^2 - \Gamma^2/16}$. As in the ideal case, we consider two other initial conditions. If the N ions are initially in the state $\prod_{j=2}^{N}|e_1\rangle|i_j\rangle$, and the cavity mode in vacuum state $|0\rangle$, we have the corresponding evolution

see equation (15) above

where $A_{1\Gamma} = \sqrt{\eta^2\Omega_1^2 - \Gamma^2/16}$. If the total system is initially in the state $\prod_{j=2,j\neq k}^{N}|e_1\rangle|g_k\rangle|i_j\rangle|0\rangle$, the evolution of the system is

see equation (16) above

with $\Omega_k' = \sqrt{\Omega_1^2 + \Omega_k^2}$ and $A_{k\Gamma} = \sqrt{\eta^2\Omega_1^2 + \eta^2\Omega_k^2 - \Gamma^2/16}$, $(k = 2, \ldots, N)$. Assuming that $\sqrt{\eta^2\Omega_1^2 - \Gamma^2/16}t = \pi$, $\sqrt{\eta^2\Omega_1^2 + \eta^2\Omega_2^2 - \Gamma^2/16}t = 6\pi$, and $\sqrt{\eta^2\Omega_1^2 + \eta^2\Omega_2^2 + \eta^2\Omega_3^2 - \Gamma^2/16}t = 10\pi$, for $N = 3$, resulting in $\cos\sqrt{\eta^2\Omega_1^2 + \eta^2\Omega_3^2 - \Gamma^2/16}t = \cos\sqrt{65}\pi = 0.9810$, we acquire an approximate three-qubit phase gate

T'_P based on equations (14), (15), and (16)

$$\begin{aligned}
&|g_1\rangle|g_2\rangle|g_3\rangle|0\rangle \to |g_1\rangle|g_2\rangle|g_3\rangle|0\rangle,\\
&|g_1\rangle|g_2\rangle|i_3\rangle|0\rangle \to |g_1\rangle|g_2\rangle|i_3\rangle|0\rangle,\\
&|g_1\rangle|i_2\rangle|g_3\rangle|0\rangle \to |g_1\rangle|i_2\rangle|g_3\rangle|0\rangle,\\
&|g_1\rangle|i_2\rangle|i_3\rangle|0\rangle \to |g_1\rangle|i_2\rangle|i_3\rangle|0\rangle,\\
&|e_1\rangle|g_2\rangle|g_3\rangle|0\rangle \to \alpha_1|e_1\rangle|g_2\rangle|g_3\rangle|0\rangle + \alpha_2|g_1\rangle|e_2\rangle|g_3\rangle|0\rangle\\
&\qquad + \alpha_3|g_1\rangle|g_2\rangle|e_3\rangle|0\rangle,\\
&|e_1\rangle|g_2\rangle|i_3\rangle|0\rangle \to \beta_1|e_1\rangle|g_2\rangle|i_3\rangle|0\rangle + \beta_2|g_1\rangle|e_2\rangle|i_3\rangle|0\rangle,\\
&|e_1\rangle|i_2\rangle|g_3\rangle|0\rangle \to \gamma_1|e_1\rangle|i_2\rangle|g_3\rangle|0\rangle\\
&\qquad + \gamma_2|g_1\rangle|i_2\rangle|e_3\rangle|0\rangle\gamma_1|e_1\rangle|i_2\rangle|g_3\rangle|0\rangle\\
&\qquad + \gamma_3|g_1\rangle|i_2\rangle|g_3\rangle|1\rangle,\\
&|e_1\rangle|i_2\rangle|i_3\rangle|0\rangle \to -\delta|e_1\rangle|i_2\rangle|i_3\rangle|0\rangle, \qquad (17)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{\Omega_1^2+\Omega_2^2+\Omega_3^2}[\Omega_1^2 e^{-\Gamma t/4} + (\Omega_2^2+\Omega_3^2)e^{-\Gamma t/2}],\\
\alpha_2 &= \frac{\Omega_1\Omega_2}{\Omega_1^2+\Omega_2^2+\Omega_3^2}[e^{-\Gamma t/4} - e^{-\Gamma t/2}],\\
\alpha_3 &= \frac{\Omega_1\Omega_3}{\Omega_1^2+\Omega_2^2+\Omega_3^2}[e^{-\Gamma t/4} - e^{-\Gamma t/2}],\\
\beta_1 &= \frac{1}{\Omega_1^2+\Omega_2^2}[\Omega_1^2 e^{-\Gamma t/4} + \Omega_2^2 e^{-\Gamma t/2}],\\
\beta_2 &= \frac{\Omega_1\Omega_2}{\Omega_1^2+\Omega_2^2}[e^{-\Gamma t/4} - e^{-\Gamma t/2}],\\
\gamma_1 &= \frac{1}{\Omega_1^2+\Omega_3^2}\{\Omega_1^2 e^{-\Gamma t/4}[\cos(\sqrt{4(l^2-m^2)+1}\pi)\\
&\quad - \frac{\Gamma}{4A_{\Gamma_3}}\sin(\sqrt{4(l^2-m^2)+1}\pi)] + \Omega_3^2 e^{-\Gamma t/2}\},\\
\gamma_2 &= \frac{\Omega_1\Omega_3}{\Omega_1^2+\Omega_3^2}\{e^{-\Gamma t/4}[\cos(\sqrt{4(l^2-m^2)+1}\pi)\\
&\quad - \frac{\Gamma}{4A_{\Gamma_3}}\sin(\sqrt{4(l^2-m^2)+1}\pi)] - e^{-\Gamma t/2}\},\\
\gamma_3 &= -i\frac{\eta\Omega_1 e^{-\Gamma t/4}}{A_{\Gamma_3}}\sin(\sqrt{4(l^2-m^2)+1}\pi),\\
\delta &= e^{-\Gamma t/4}, \quad t = \pi/A_{\kappa 1} \simeq \pi/(\eta\Omega_1). \qquad (18)
\end{aligned}$$

When $\Gamma = \eta\Omega_1/25$, $\alpha_1, \alpha_2, \alpha_3$, β_1, β_2, $\gamma_1, \gamma_2, \gamma_3$ and δ are, respectively, equal to 0.9394, 0.0018, 0.0024, 0.9399, 0.0049, 0.9392, 0.0007, $-0.0234i$ and 0.9691, leading to an approximate three-qubit phase gate T'_P.

$$\begin{aligned}
T'_P|g_1\rangle|g_2\rangle|g_3\rangle|0\rangle &= |g_1\rangle|g_2\rangle|g_3\rangle|0\rangle,\\
T'_P|g_1\rangle|g_2\rangle|i_3\rangle|0\rangle &= |g_1\rangle|g_2\rangle|i_3\rangle|0\rangle,\\
T'_P|g_1\rangle|i_2\rangle|g_3\rangle|0\rangle &= |g_1\rangle|i_2\rangle|g_3\rangle|0\rangle,\\
T'_P|g_1\rangle|i_2\rangle|i_3\rangle|0\rangle &= |g_1\rangle|i_2\rangle|i_3\rangle|0\rangle,\\
T'_P|e_1\rangle|g_2\rangle|g_3\rangle|0\rangle &\simeq 0.9394|e_1\rangle|g_2\rangle|g_3\rangle|0\rangle,\\
T'_P|e_1\rangle|g_2\rangle|i_3\rangle|0\rangle &\simeq 0.9399|e_1\rangle|g_2\rangle|i_3\rangle|0\rangle,\\
T'_P|e_1\rangle|i_2\rangle|g_3\rangle|0\rangle &\simeq 0.9392|e_1\rangle|i_2\rangle|g_3\rangle|0\rangle,\\
T'_P|e_1\rangle|i_2\rangle|i_3\rangle|0\rangle &\simeq -0.9691|e_1\rangle|i_2\rangle|i_3\rangle|0\rangle, \qquad (19)
\end{aligned}$$

where we neglect the very small terms with a relative error order of 10^{-4}.Therefore, we can obtain an approximate Toffoli gate for $\Gamma/\eta\Omega_1 = 1/25$, by combining the phase gate T'_P and the Hadamard transform H_3 on the third qubit,

$$\begin{aligned}
T' &= H_3 T'_P H_3\\
&= \begin{bmatrix}
1\,0\,0\,0 & 0 & 0 & 0 & 0\\
0\,1\,0\,0 & 0 & 0 & 0 & 0\\
0\,0\,1\,0 & 0 & 0 & 0 & 0\\
0\,0\,0\,1 & 0 & 0 & 0 & 0\\
0\,0\,0\,0 & 0.9396 & -0.0003 & 0 & 0\\
0\,0\,0\,0 & -0.0003 & 0.9396 & 0 & 0\\
0\,0\,0\,0 & 0 & 0 & -0.0149 & 0.9451\\
0\,0\,0\,0 & 0 & 0 & 0.9451 & -0.0149
\end{bmatrix}\\
&\simeq \begin{bmatrix}
1\,0\,0\,0 & 0 & 0 & 0 & 0\\
0\,1\,0\,0 & 0 & 0 & 0 & 0\\
0\,0\,1\,0 & 0 & 0 & 0 & 0\\
0\,0\,0\,1 & 0 & 0 & 0 & 0\\
0\,0\,0\,0 & 0.9396 & 0 & 0 & 0\\
0\,0\,0\,0 & 0 & 0.9396 & 0 & 0\\
0\,0\,0\,0 & 0 & 0 & 0 & 0.9451\\
0\,0\,0\,0 & 0 & 0 & 0.9451 & 0
\end{bmatrix}. \qquad (20)
\end{aligned}$$

To check the fidelity of our proposed gate, we use equations (11) and (19) to prepare an entangled state from an initial state,

$$|\varphi_0\rangle = \frac{1}{2\sqrt{2}}(|g_1\rangle + |e_1\rangle)(|g_2\rangle + |i_2\rangle)(|g_3\rangle + |i_3\rangle). \qquad (21)$$

We have

$$\begin{aligned}
|\varphi_{ideal}\rangle = \frac{1}{2\sqrt{2}}&[|g_1\rangle\,(|g_2\rangle\,|g_3\rangle + |g_2\rangle\,|i_3\rangle + |i_2\rangle\,|g_3\rangle\\
&+ |i_2\rangle\,|i_3\rangle) + |e_1\rangle\,(|g_2\rangle\,|g_3\rangle + |g_2\rangle\,|i_3\rangle\\
&+ |i_2\rangle\,|g_3\rangle - |i_2\rangle\,|i_3\rangle)], \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
|\varphi_{decay}\rangle_{\Gamma=\eta\Omega_1/25} = \frac{1}{\sqrt{7.587}}&[|g_1\rangle(|g_2\rangle|g_3\rangle+|g_2\rangle|i_3\rangle+|i_2\rangle|g_3\rangle\\
&+ |i_2\rangle|i_3\rangle) + |e_1\rangle(0.9394|g_2\rangle|g_3\rangle\\
&+ 0.9399|g_2\rangle|i_3\rangle + 0.9392|i_2\rangle|g_3\rangle\\
&- 0.9691|i_2\rangle|i_3\rangle)]. \qquad (23)
\end{aligned}$$

So the fidelity and the corresponding success probability are $F = |\langle\varphi_{ideal}|\,\varphi_{spon}\rangle|^2 = 0.9996$ and $P = 0.9484$, for $\Gamma = \eta\Omega_1/25$. Our proposed gate contains some imperfection such as the spontaneous atomic emission. It obviously has a high fidelity and high-success probability. This should be compared with previous schemes in reference [8] which have been used to study two-qubit phase gates based on a cavity QED system. It is noted that the approximation in the ideal case produces a deviation of the order of 10^{-5} on the fidelity and success probability if we use the ideal three-qubit phase gate to prepare the entangled state.

Now we will briefly discuss the experimental feasibility of our proposed scheme. We may employ $^{40}Ca^+$ ions for our proposal with $|i_j\rangle$, $|g_j\rangle$, and $|e_j\rangle$ given by the states

$S_{1/2}(m_j = 1/2)$, $S_{1/2}(m_j = -1/2)$ and $D_{5/2}(m_j = -1/2)$, respectively [9]. From the derivation of [9], the possibility exists of reaching the Lamb-Dicke limit and so obtain different precise couplings for the three ions confined in a linear trap, although addressing any more than two trapped ions with precision has not been reported so far. Apart from the general requirements of the LB quantum gate proposal in reference [10], the new requirement is that the lasers should have different specific intensities, satisfying the gate gate condition $\Omega_1 : \Omega_2 : \Omega_3 = 1 : \sqrt{35} : 8$. Since the coupling strengths between the three ions and the laser fields must be precisely controlled, the laser power must be stable to within about $\pm 0.5\%$ [10] and the corresponding positions of the ions in the trap must be also specifically located, which is challenging. However, we believe that our scheme is within the reach of current trapped ion techniques.

As far as we know, our proposal gives the simplest implementation of a Toffoli gate so far in a trapped ion system. Although there is the spontaneous atomic emission involved, we have shown that the achievable fidelity and success probability are high enough to meet the requirement for quantum information processing and quantum computation. For example, our scheme may be used for a three-qubit Grover search to label the target qubit. So far, a two-qubit Grover search has been proposed in cavity QED [11–13] and in ion traps [14], and recently was carried out in an ion trap [15]. But no experimental report for a three-qubit Grover search has been found.

In conclusion, we have proposed a feasible scheme for the realization of a three-qubit Toffoli gate based on a single resonant interaction of the trapped ions with laser fields. The scheme does not require two-qubit controlled-NOT gates but uses a three-qubit phase gate and two Hadamard gates to construct the Toffoli gate. It also does not require using the vibrational mode as the data bus and only requires one resonant interaction. The required interaction time is very short due to the resonant interaction without involving many laser fields [16]. In fact, a fast two-qubit quantum gate between laser trapped cold ions has been addressed by Jonathan et al. in reference [10], which is based on the ac stark shift induced by the laser light resonant with the ionic transition frequency of two-level ions and requires specific laser intensities. In that scheme, six pulses are required for the implementation of a C-NOT gate, consisting of three one-qubit and three two-qubit operations. The required duration is about $(2/\eta\nu) \times 3 = 60/\nu$, where $\eta = 0.1$. The realization of the Toffoli gate with the method in reference [10] requires two-C-NOT gate operations and the corresponding time is about $120/\nu$. In contrast to reference [10], three three-level ions are considered in our scheme and the computational basis is composed of $\{|g_1\rangle|g_2\rangle|g_3\rangle, |g_1\rangle|g_2\rangle|i_3\rangle, |g_1\rangle|i_2\rangle|g_3\rangle, |g_1\rangle|i_2\rangle|i_3\rangle, |e_1\rangle|g_2\rangle|g_3\rangle, |e_1\rangle|g_2\rangle|i_3\rangle, |e_1\rangle|i_2\rangle|g_3\rangle, |e_1\rangle|i_2\rangle|i_3\rangle\}$. For $\eta = 0.1$ and $\Omega = \nu/2$ [10], the duration of implementation of the Toffoli gate is about $20\pi/\nu$, which is shorter than that in reference [10] and the number of required pulses is also fewer than that in reference [10]. Furthermore, dissipation due to spontaneous ionic emission is involved in our scheme, which was not considered in reference [10]. We have also paid attention to the review paper on ion trap quantum computing by Šašura and Bužek [17], where a summary on the quantum gates, in which dissipation had not been considered, was given. Both situations in the presence and in the absence of the spontaneous atomic emission have been considered in our scheme. Compared with the previous schemes for two-qubit quantum gates [5,6,8,10,12,17], our scheme can simplify the operational steps in the realization of the Toffoli gate with high efficiency. We believe that the present proposal should be useful for advances in trapped ion experiments and is within the reach of current trapped ion techniques.

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References

1. P.W. Shor, in *Proceedings of the 35th Annual Symposium on Foudations of Computer Science* (IEEE Computer Society, Los Alamitos, CA, 1994), p. 116
2. L.K. Grover, Phys. Rev. Lett. **79**, 4709 (1997)
3. T. Sleator, H. Weinfurter, Phys. Rev. Lett. **74**, 4087 (1995)
4. T. Toffoli, in *Automata Languages and Programming, Seventh Colloquium*, edited by J.W. de Bakker, J. van Leeuwen, Lectures Notes in Computer Science (Springer, New York, 1980), Vol. 84
5. M.D. Price et al., Phys. Rev. A **60**, 2777 (1999); D.G. Gory et al., Phys. Rev. Lett. **81**, 2152 (1998)
6. L.-M. Duan, B. Wang, H.J. Kimble, Phys. Rev. A **72**, 032333 (2005)
7. A. Gàbris, G.S. Agarwal, Phys. Rev. A **71**, 052316 (2005)
8. J. Pachos, H. Walther, Phys. Rev. Lett. **89**, 187903 (2002); S.B. Zheng, Phys. Rev. A **71**, 062335 (2005)
9. C.F. Roos et al., Science **304**, 1478 (2004)
10. D. Jonathan, M.B. Plenio, P.L. Kinght, Phys. Rev. A **62**, 042307 (2002)
11. M. Feng, X.G. Wang, J. Opt. B: Quantum Semiclass. **4**, 283 (2002)
12. F. Yamaguchi et al., Phys. Rev. A **66**, 010302(R) (2002)
13. Z.J. Deng, M. Feng, K.L. Gao, Phys. Rev. A **72**, 034306 (2005)
14. M. Feng, Phys. Rev. A **63**, 052308 (2001)
15. K.A. Brickman et al., Phys. Rev. A **72**, 050306(R) (2005)
16. J.I. Cirac, P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995); J.J. Garcia, P. Zoller, J.I. Cirac, Phys. Rev. Lett. **91**, 157901 (2003); M. Šašura, A.M. Steane, Phys. Rev. A **67**, 062318 (2003); S.B. Zheng, Phys. Rev. Lett. **90**, 217901 (2003)
17. M. Šašura, V. Bužek, J. Mod. Opt. **49**, 1593 (2002)